

# STABILITY ESTIMATE FOR AN INVERSE PROBLEM FOR THE SCHRÖDINGER EQUATION IN A MAGNETIC FIELD WITH TIME-DEPENDENT COEFFICIENT

IBTISSEM BEN AÏCHA

**ABSTRACT.** We study the stability issue in the inverse problem of determining the magnetic field and the time-dependent electric potential appearing in the Schrödinger equation, from boundary observations. We prove in dimension 3 or greater, that the knowledge of the Dirichlet-to-Neumann map stably determines the magnetic field and the electric potential.

**keywords:** Stability estimates, Schrödinger equation, magnetic field, time-dependent electric potential, Dirichlet-to-Neumann map.

## 1. INTRODUCTION

**1.1. Statement of the problem.** The present paper deals with the inverse problem of determining the magnetic field and the time-dependent electric potential in the magnetic Schrödinger equation from the knowledge of boundary observations. Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded and simply connected domain with  $C^\infty$  boundary  $\Gamma$ . We denote by  $\Delta_A$  the Laplace operator associated to the real valued magnetic potential  $A \in C^3(\Omega)$  which is defined by

$$\Delta_A = \sum_{j=1}^n (\partial_j + ia_j)^2 = \Delta + 2iA \cdot \nabla + i \operatorname{div}(A) - |A|^2.$$

Given  $T > 0$ , we denote by  $Q = \Omega \times (0, T)$  and  $\Sigma = \Gamma \times (0, T)$ . We consider the following initial boundary problem for the Schrödinger equation

$$(1.1) \quad \begin{cases} (i\partial_t + \Delta_A + q(x, t))u = 0, & \text{in } Q, \\ u(., 0) = u_0, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma, \end{cases}$$

where the real valued bounded function  $q \in W^{2,\infty}(0, T; W^{1,\infty}(\Omega))$  is the electric potential. We define the Dirichlet-to-Neumann map associated to the magnetic Schrödinger equation (1.1) as

$$\begin{aligned} \Lambda_{A,q} : H^2(\Omega) \times H^{2,1}(\Sigma) &\longrightarrow H^1(\Omega) \times L^2(\Sigma) \\ (u_0, f) &\longmapsto (u(., T), (\partial_\nu + iA \cdot \nu)u), \end{aligned}$$

where  $\nu(x)$  denotes the unit outward normal to  $\Gamma$  at  $x$ , and  $\partial_\nu u$  stands for  $\nabla u \cdot \nu$ . Here  $H^{2,1}(\Sigma)$  is a Sobolev space we shall make precise below. We aim to know whether the knowledge of the Dirichlet-to-Neumann map  $\Lambda_{A,q}$  can uniquely determine the magnetic and the electric potentials.

The problem of recovering coefficients in the magnetic Schrödinger equation was treated by many authors. In [5], Bellassoued and Choulli considered the problem of recovering the magnetic potential  $A$  from the knowledge of the Dirichlet-to-Neumann map  $\Lambda_A(f) = (\partial_\nu + i\nu \cdot A)u$  for  $f \in L^2(\Sigma)$ , associated to the Schrödinger equation with zero initial data. As it was noted in [21], the Dirichlet-to-Neumann map  $\Lambda_A$  is

invariant under the gauge transformation of the magnetic potential. Namely, given  $\varphi \in C^1(\overline{\Omega})$  such that  $\varphi|_{\Gamma} = 0$ , we have

$$(1.2) \quad e^{-i\varphi} \Delta_A e^{i\varphi} = \Delta_{A+\nabla\varphi}, \quad e^{-i\varphi} \Lambda_A e^{i\varphi} = \Lambda_{A+\nabla\varphi},$$

and  $\Lambda_A = \Lambda_{A+\nabla\varphi}$ . Therefore, the magnetic potential  $A$  can not be uniquely determined by the Dirichlet-to-Neumann map  $\Lambda_A$ . In geometric terms, the magnetic potential  $A$  defines the connection given by the one form  $\alpha_A = \sum_{j=1}^n a_j dx_j$ . The non uniqueness manifested in (1.2) says that the best one can hope to recover from the Dirichlet-to-Neumann map is the 2-form

$$d\alpha_A = \sum_{i,j=1}^n \left( \frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} \right) dx_j \wedge dx_i,$$

called the magnetic field. Bellassoued and Choulli proved in dimension  $n \geq 2$  that the knowledge of the Dirichlet-to-Neumann map  $\Lambda_A$  Hölder stably determines the magnetic field  $d\alpha_A$ .

In the presence of a time-independent electric potential, the inverse problem of determining the magnetic field  $d\alpha_A$  and the electric potential  $q$  from boundary observations was first considered by Sun [24], in the case  $n \geq 3$ . He showed that  $d\alpha_A$  and  $q$  can be uniquely determined when  $A \in W^{2,\infty}$ ,  $q \in L^\infty$  and  $d\alpha_A$  is small in the  $L^\infty$  norm. In [9], Benjoud studied the inverse problem of recovering the magnetic field  $d\alpha_A$  and the electric potential  $q$  from the knowledge of the Dirichlet-to-Neumann map. Assuming that the potentials are known in a neighborhood of the boundary, she proved a stability estimate with respect to arbitrary partial boundary observations.

In the Riemannian case, Bellassoued [2] proved recently a Hölder-type stability estimate in the recovery of the magnetic field  $d\alpha_A$  and the time-independent electric potential  $q$  from the knowledge of the Dirichlet-to-Neumann map associated to the Shrödinger equation with zero initial data. In the absence of the magnetic potential  $A$ , the problem of recovering the electric potential  $q$  on a compact Riemannian manifold was solved by Bellassoued and Dos Santos Ferreira [7].

In recent years significant progress have been made in the recovery of time-dependent and time-independent coefficients appearing in hyperbolic equations, see for instance [6, 13, 23]. We also refer to the work of Bellassoued and Benjoud [4] in which they prove that the Dirichlet-to-Neumann map determines uniquely the magnetic field in a magnetic wave equation. In [22], Eskin proved that the Dirichlet-to-Neumann map uniquely determines coefficients depending analytically on the time variable. In [18], Stefanov proved that the time-dependent potential  $q$  appearing in the wave equation is uniquely determined from the knowledge of scattering data. In [14], Ramm and Sjöstrand proved a uniqueness result in recovering the time-dependent potential  $q$  from the Dirichlet-to-Neumann map, on the infinite time-space cylindrical domain  $\mathbb{R}_t \times \Omega$ . As for stability results, we refer to Salazar [15], Waters [27], Ben Aïcha [8] and Kian [20].

The problem of determining time-dependent electromagnetic potentials appearing in a Schrödinger equation was treated by Eskin [21]. Using a geometric optics construction, he prove the uniqueness for this problem in domains with obstacles. In unbounded domains and in the absence of the magnetic potential, Choulli, Kian and Soccorsi [10] treated the problem of recovering the time-dependent scalar potential  $q$  appearing in the Schrödinger equation from boundary observations. Assuming that the domain is a 1-periodic cylindrical waveguide, they proved logarithmic stability for this problem.

In the present paper, we address the uniqueness and the stability issues in the inverse problem of recovering the magnetic field  $d\alpha_A$  and the time-dependent potential  $q$  in the dynamical Schrödinger equation,

from the knowledge of the operator  $\Lambda_{A,q}$ . By means of techniques used in [2, 9], we prove a "log-type" stability estimate in the recovery of the magnetic field and a "log-log-log-type" stability inequality in the determination of the time-dependent electric potential.

From a physical view point, our inverse problem consists in determining the magnetic field  $d\alpha_A$  induced by the magnetic potential  $A$ , and the electric potential  $q$  of an inhomogeneous medium by probing it with disturbances generated on the boundary. Here we assume that the medium is quiet initially and  $f$  denotes the disturbance used to probe the medium. Our data are the response  $(\partial_\nu + iA \cdot \nu)u$  performed on the boundary  $\Sigma$ , and the measurement  $u(\cdot, T)$ , for different choices of  $f$  and for all possible initial data  $u_0$ .

**1.2. Well-posedness of the magnetic Schrödinger equation and main results.** In order to state our main results, we need the following existence and uniqueness result. To this end, we introduce the following Sobolev space

$$H^{2,1}(\Sigma) = H^2(0, T; L^2(\Gamma)) \cap L^2(0, T; H^1(\Gamma)),$$

equipped with the norm

$$\|f\|_{H^{2,1}(\Sigma)} = \|f\|_{H^2(0,T;L^2(\Gamma))} + \|f\|_{L^2(0,T;H^1(\Gamma))},$$

and we set

$$H_0^{2,1}(\Sigma) = \{f \in H^{2,1}(\Sigma), f(\cdot, 0) = \partial_t f(\cdot, 0) = 0\}.$$

Then we have the following theorem.

**Theorem 1.1.** *Let  $T > 0$  and let  $q \in W^{1,\infty}(Q)$ ,  $A \in C^1(\Omega)$  and  $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ . Suppose that  $f \in H_0^{2,1}(\Sigma)$ . Then, there exists a unique solution  $u \in C(0, T; H^1(\Omega))$  of the Shrödinger equation (1.1). Furthermore, we have  $\partial_\nu u \in L^2(\Sigma)$  and there exists a constant  $C > 0$  such that*

$$\|u(\cdot, t)\|_{H^1(\Omega)} + \|\partial_\nu u\|_{L^2(\Sigma)} \leq C (\|u_0\|_{H^2(\Omega)} + \|f\|_{H^{2,1}(\Sigma)}).$$

As a corollary, the Dirichlet-to-Neumann map  $\Lambda_{A,q}$  is bounded from  $H^2(\Omega) \times H^{2,1}(\Sigma)$  to  $H^1(\Omega) \times L^2(\Sigma)$ . The proof of Theorem 1.1 is given in Appendix A.

In order to express the main results of this article, we first define the following admissible sets of unknown coefficients  $A$  and  $q$ : for  $\varepsilon > 0$ ,  $M > 0$ , we set

$$\mathcal{A}_\varepsilon = \{A \in C^3(\Omega), \|A\|_{W^{3,\infty}(\Omega)} \leq \varepsilon, A_1 = A_2 \text{ in } \Gamma\},$$

$$\mathcal{Q}_M = \{q \in \mathcal{X} = W^{2,\infty}(0, T; W^{1,\infty}(\Omega)), \|q\|_{\mathcal{X}} \leq M, q_1 = q_2 \text{ in } \Gamma\}.$$

Our first main result claims stable determination of the magnetic field  $d\alpha_A$ , from full boundary measurement  $\Lambda_{A,q}$  on the cylindrical domain  $Q$ .

**Theorem 1.2.** *Let  $\alpha > \frac{n}{2} + 1$ . Let  $q_i \in \mathcal{Q}_M$ ,  $A_i \in \mathcal{A}_\varepsilon$ , such that  $\|A_i\|_{H^\alpha(\Omega)} \leq M$ , for  $i = 1, 2$ . Then, there exist three constants  $C > 0$  and  $\mu, s \in (0, 1)$ , such that we have*

$$\|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} \leq C \left( \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\|^{1/2} + |\log \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\||^{-\mu} \right)^s.$$

Here  $C$  depends only on  $\Omega$ ,  $\varepsilon$ ,  $M$  and  $T$ .

Next, assuming that the magnetic potential  $A$  is divergence free, we can stably retrieve the electric potential.

**Theorem 1.3.** *Let  $q_i \in \mathcal{Q}_M$ ,  $A_i \in \mathcal{A}_\varepsilon$ , for  $i = 1, 2$ . Assume that  $\operatorname{div} A_i = 0$ . Then there exist three constants  $C > 0$ , and  $m, \mu \in (0, 1)$ , such that we have*

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leq C \Phi_m(\|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\|),$$

where

$$\Phi_m(\|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\|) = \begin{cases} |\log |\log |\log \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\|^\mu||^{-1} & \text{if } \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| < m, \\ \frac{1}{m} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| & \text{if } \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| \geq m. \end{cases}$$

Here  $C$  depends on  $\Omega$ ,  $M$ ,  $\varepsilon$  and  $T$ .

The text is organized as follows. Section 2 is devoted to the construction of special geometrical optics solutions to the Shrödinger equation (1.1). Using these particular solutions, we establish in sections 3 and 4, two stability estimates for the magnetic field and the electric potential. In Appendix A, we develop the proof of Theorem 1.1. Appendix B contains the proof of several technical results used in the derivation of the main results.

## 2. PRELIMINARIES AND GEOMETRICAL OPTICS SOLUTIONS

The present section is devoted to the construction of suitable geometrical optics solutions, which are key ingredients in the proof of our main results. We start by collecting several known lemmas from [17, 19].

**2.1. Preliminaries.** Let  $\omega = \omega_{\Re} + i\omega_{\Im}$  be a vector with  $\omega_{\Re}, \omega_{\Im} \in \mathbb{S}^{n-1}$ , and  $\omega_{\Re} \cdot \omega_{\Im} = 0$ . We shall see that the differential operator  $N_\omega = \omega \cdot \nabla$  is invertible and we have

$$N_\omega^{-1}(g)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \left( \frac{\hat{g}(\xi)}{\omega \cdot \xi} \right) d\xi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{y_1 + iy_2} g(x - y_1 \omega_{\Re} - y_2 \omega_{\Im}) dy_1 dy_2.$$

Notice that the differential operator  $\bar{\partial}$  corresponds to  $N_\omega$  with  $\omega = (0, 1)$ .

**Lemma 2.1.** *Let  $r > 0$ ,  $k > 0$  and let  $g \in W^{k, \infty}(\mathbb{R}^n)$  be such that  $\operatorname{Supp} g \subseteq B(0, r) = \{x \in \mathbb{R}^n, |x| \leq r\}$ . Then the function  $\phi = N_\omega^{-1}(g) \in W^{k, \infty}(\mathbb{R}^n)$  solves  $N_\omega(\phi) = g$ , and satisfies the estimate*

$$\|\phi\|_{W^{k, \infty}(\mathbb{R}^n)} \leq C \|g\|_{W^{k, \infty}(\mathbb{R}^n)},$$

where  $C$  is a positive constant depending only on  $r$ .

We recall from [16], the following technical result.

**Lemma 2.2.** *Let  $A \in C_c(\mathbb{R}^n)$ ,  $\xi \in \mathbb{R}^n$ , and  $\omega = \omega_{\Re} + i\omega_{\Im}$  with  $\omega_{\Re}, \omega_{\Im} \in \mathbb{S}^{n-1}$  and  $\omega_{\Re} \cdot \omega_{\Im} = \omega_{\Re} \cdot \xi = \omega_{\Im} \cdot \xi = 0$ . Then we have the following identity*

$$\int_{\mathbb{R}^n} \omega \cdot A(x) e^{iN_\omega^{-1}(-\omega \cdot A)(x)} e^{i\xi \cdot x} dx = \int_{\mathbb{R}^n} \omega \cdot A(x) e^{i\xi \cdot x} dx.$$

**2.2. Geometrical optics solutions.** In this section, we build special solutions to the magnetic Schrödinger equation (1.1), inspired by techniques used in elliptic problems. For this purpose, we consider a vector  $\omega = \omega_{\Re} + i \omega_{\Im}$ , such that  $\omega_{\Re}, \omega_{\Im} \in \mathbb{S}^{n-1}$  and  $\omega_{\Re} \cdot \omega_{\Im} = 0$ . For  $\sigma > 1$ , we define the complex variable  $\rho$  as follows

$$(2.3) \quad \rho = \sigma\omega + y,$$

where  $y \in B(0, 1)$  is fixed and independent of  $\sigma$ . In what follows,  $P(D)$  denotes a differential operator with constant coefficients:

$$P(D) = \sum_{|\alpha| \leq m} a_{\alpha} D^{\alpha}, \quad D = -i(\partial_t, \partial_x).$$

We associate to the operator  $P(D)$  its symbol  $p(\xi, \tau)$  defined by

$$p(\xi, \tau) = \sum_{|\alpha| \leq m} a_{\alpha}(\xi, \tau)^{\alpha}, \quad (\xi, \tau) \in \mathbb{R}^{n+1}.$$

Moreover, we set

$$\tilde{p}(\xi, \tau) = \left( \sum_{\beta \in \mathbb{N}} \sum_{\alpha \in \mathbb{N}^n} |\partial_{\tau}^{\beta} \partial_{\xi}^{\alpha} p(\xi, \tau)|^2 \right)^{\frac{1}{2}}, \quad (\xi, \tau) \in \mathbb{R}^{n+1},$$

and introduce the operators

$$\Delta_{\rho} = \Delta - 2i\rho \cdot \nabla \quad \text{and} \quad \nabla_{\rho} = \nabla - i\rho.$$

We turn now to building particular solutions to the magnetic Schrödinger equation. We proceed with a succession of lemmas. The first result is inspired by Hörmander [11] (see Appendix B).

**Lemma 2.3.** *Let  $P \neq 0$  be an operator. There exists a linear operator  $E \in \mathcal{L}(L^2(0, T; H^1(\Omega)))$ , such that:*

$$P(D)Ef = f, \quad \text{for any } f \in L^2(0, T; H^1(\Omega)).$$

Moreover, for any linear operator  $S$  with constant coefficients such that  $\frac{|S(\xi, \tau)|}{\tilde{p}(\xi, \tau)}$  is bounded in  $\mathbb{R}^{n+1}$ , we have the following estimate

$$(2.4) \quad \|S(D)Ef\|_{L^2(0, T; H^1(\Omega))} \leq C \sup_{\mathbb{R}^{n+1}} \frac{|S(\xi, \tau)|}{\tilde{p}(\xi, \tau)} \|f\|_{L^2(0, T; H^1(\Omega))}.$$

Here  $C$  depends only on the degree of  $P$ ,  $\Omega$  and  $T$ .

**Lemma 2.4.** *There exists a bounded operator  $E_{\rho} : L^2(0, T; H^1(\Omega)) \longrightarrow L^2(0, T; H^2(\Omega))$  such that*

$$P_{\rho}(D)E_{\rho}f = (i\partial_t + \Delta_{\rho})E_{\rho}f = f \quad \text{for any } f \in L^2(0, T; H^1(\Omega)).$$

Moreover, there exists a constant  $C(\Omega, T) > 0$  such that

$$(2.5) \quad \|E_{\rho}f\|_{L^2(0, T; H^k(\Omega))} \leq \frac{C}{\sigma^{2-k}} \|f\|_{L^2(0, T; H^1(\Omega))}, \quad k = 1, 2.$$

*Proof.* From Lemma 2.3, we deduce the existence of a linear operator  $E_{\rho} \in \mathcal{L}(L^2(0, T; H^1(\Omega)))$  such that  $P_{\rho}(D)E_{\rho}f = f$ . Moreover, since  $|\tilde{p}_{\rho}(\xi, \tau)| > \sigma$ , we get from (2.4)

$$(2.6) \quad \|E_{\rho}f\|_{L^2(0, T; H^1(\Omega))} \leq \frac{C}{\sigma} \|f\|_{L^2(0, T; H^1(\Omega))}.$$

Similarly, since  $\frac{|\xi|}{\tilde{p}_\rho(\xi, \tau)}$  is bounded on  $\mathbb{R}^{n+1}$ , we get

$$\|\nabla E_\rho f\|_{L^2(0,T;H^1(\Omega))} \leq C \|f\|_{L^2(0,T;H^1(\Omega))}.$$

From this and (2.6) we see that  $E_\rho$  is bounded from  $L^2(0, T; H^1(\Omega))$  into  $L^2(0, T; H^2(\Omega))$ .  $\square$

Let us now deduce the coming statement from the above lemma.

**Lemma 2.5.** *There exists  $\varepsilon > 0$  such that for all  $A \in W^{1,\infty}(\Omega)$  obeying  $\|A\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$ , we may build a bounded operator  $F_\rho : L^2(0, T; H^1(\Omega)) \longrightarrow L^2(0, T; H^2(\Omega))$  such that:*

$$(2.7) \quad (i\partial_t + \Delta_\rho + 2iA \cdot \nabla) F_\rho f = f, \quad \text{for any } f \in L^2(0, T; H^1(\Omega)).$$

Moreover, there exists a constant  $C(\Omega, T) > 0$  such that

$$(2.8) \quad \|F_\rho f\|_{L^2(0,T;H^k(\Omega))} \leq \frac{C}{\sigma^{2-k}} \|f\|_{L^2(0,T;H^1(\Omega))}, \quad k = 1, 2.$$

*Proof.* Let  $f \in L^2(0, T; H^1(\Omega))$ . We start by introducing the following operator

$$\begin{aligned} S_\rho : L^2(0, T; H^2(\Omega)) &\longrightarrow L^2(0, T; H^2(\Omega)) \\ g &\longmapsto E_\rho(-2iA \cdot \nabla g + f). \end{aligned}$$

Since  $\|A\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$ , we deduce from (2.5) with  $k = 2$  that

$$(2.9) \quad \|S_\rho(h) - S_\rho(g)\|_{L^2(0,T;H^2(\Omega))} \leq C\varepsilon \|h - g\|_{L^2(0,T;H^2(\Omega))},$$

for any  $h, g \in L^2(0, T; H^2(\Omega))$ . Thus,  $S_\rho$  is a contraction from  $L^2(0, T; H^2(\Omega))$  into  $L^2(0, T; H^2(\Omega))$  for  $\varepsilon$  small enough. Then,  $S_\rho$  admits a unique fixed point  $g \in L^2(0, T; H^2(\Omega))$ . Put  $F_\rho f = g$ . It is clear that  $F_\rho f$  is a solution to (2.7). Then, taking into account the identity  $S_\rho F_\rho f = E_\rho(-2iA \cdot \nabla F_\rho f + f)$  and the estimate (2.9), we get

$$\begin{aligned} \|F_\rho f\|_{L^2(0,T;H^2(\Omega))} &= \|S_\rho F_\rho f - S_\rho(0)\|_{L^2(0,T;H^2(\Omega))} + \|S_\rho(0)\|_{L^2(0,T;H^2(\Omega))} \\ &\leq C\varepsilon \|F_\rho f\|_{L^2(0,T;H^2(\Omega))} + \|E_\rho f\|_{L^2(0,T;H^2(\Omega))}. \end{aligned}$$

From this and (2.5) with  $k = 2$ , we end up getting for  $\varepsilon$  small enough

$$(2.10) \quad \|F_\rho f\|_{L^2(0,T;H^2(\Omega))} \leq C \|f\|_{L^2(0,T;H^1(\Omega))}.$$

This being said, it remains to show (2.8) for  $k = 1$ . To see this, we notice from (2.5) with  $k = 1$  that

$$\begin{aligned} \|F_\rho f\|_{L^2(0,T;H^1(\Omega))} &\leq \|E_\rho(-2iA \cdot \nabla F_\rho f + f)\|_{L^2(0,T;H^1(\Omega))} \\ &\leq \frac{C}{\sigma} (\varepsilon \|F_\rho f\|_{L^2(0,T;H^2(\Omega))} + \|f\|_{L^2(0,T;H^1(\Omega))}). \end{aligned}$$

Then the estimate (2.8) for  $k = 1$  follows readily from this and (2.10).  $\square$

**Lemma 2.6.** *There exists  $\varepsilon > 0$  such that for all  $A \in W^{1,\infty}(\Omega)$  obeying  $\|A\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$ , we may build a bounded operator  $G_\rho : L^2(0, T; H^1(\Omega)) \longrightarrow L^2(0, T; H^2(\Omega))$  such that:*

$$(2.11) \quad (i\partial_t + \Delta_\rho + 2iA \cdot \nabla_\rho) G_\rho f = f \quad \text{for any } f \in L^2(0, T; H^1(\Omega)).$$

Moreover, there exists a constant  $C(\Omega, T) > 0$  such that

$$(2.12) \quad \|G_\rho f\|_{L^2(0,T;H^k(\Omega))} \leq \frac{C}{\sigma^{2-k}} \|f\|_{L^2(0,T;H^1(\Omega))}, \quad k = 1, 2.$$

*Proof.* Let  $f \in L^2(0, T; H^1(\Omega))$ . We introduce the following operator

$$\begin{aligned} R_\rho : L^2(0, T; H^1(\Omega)) &\longrightarrow L^2(0, T; H^1(\Omega)) \\ g &\longmapsto F_\rho(-2\rho \cdot Ag + f) \end{aligned}$$

From (2.3), we see that  $|\rho| < 3\sigma$ . Thus, arguing as in the proof of Lemma 2.5, we prove the existence of a unique solution  $G_\rho f = g$  to the equation (2.11). Moreover there exists a positive constants  $C > 0$  such that we have

$$(2.13) \quad \|u\|_{L^2(0, T; H^1(\Omega))} \leq \frac{C}{\sigma} \|f\|_{L^2(0, T; H^1(\Omega))}.$$

Further, combining the definition of  $R_\rho$  with (2.7) we deduce (2.12) for  $k = 2$ .  $\square$

Armed with lemma 2.6, we are now in position to establish the main result of this section, which can be stated as follows

**Lemma 2.7.** *Let  $M > 0$ ,  $\varepsilon > 0$ ,  $\omega \in \mathbb{S}^{n-1}$  and  $A \in \mathcal{A}_\varepsilon$  satisfy  $\|A\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$ . Put  $\phi = N_\omega^{-1}(-\omega \cdot A)$ . Then, for all  $\sigma \geq \sigma_0 > 0$  the magnetic Schrödinger equation*

$$(2.14) \quad (i\partial_t + \Delta_A + q(x, t))u(x, t) = 0, \quad \text{in } Q$$

*admits a solution  $u \in H^2(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ , of the form*

$$(2.15) \quad u(x, t) = e^{-i((\rho \cdot \rho)t + x \cdot \rho)} (e^{i\phi(x)} + w(x, t)),$$

*in such a way that*

$$(2.16) \quad \omega \cdot \nabla \phi(x) = -\omega \cdot A(x), \quad x \in \mathbb{R}^n.$$

*Moreover,  $w \in H^2(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$  satisfies*

$$(2.17) \quad \sigma \|w\|_{H^2(0, T; H^1(\Omega))} + \|w\|_{L^2(0, T; H^2(\Omega))} \leq C,$$

*where the constants  $C$  and  $\sigma_0$  depend only on  $\Omega, T$  and  $M$ .*

Here we extended  $A$  by zero outside  $\Omega$ .

*Proof.* To prove our lemma, it is enough to show that  $w \in H^2(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$  satisfies the estimate (2.17). Substituting (2.15) into the equation (2.14), one gets

$$\begin{aligned} \left( i\partial_t + \Delta_\rho + 2iA(x) \cdot \nabla_\rho + h(x, t) \right) w(x, t) = & -e^{i\phi(x)} \left( i\Delta\phi(x) - |\nabla\phi(x)|^2 + 2\sigma\omega \cdot \nabla\phi(x) + 2\sigma\omega \cdot A(x) \right. \\ & \left. + 2y \cdot \nabla\phi(x) + 2A(x) \cdot y - 2A(x) \cdot \nabla\phi(x) + h(x, t) \right), \end{aligned}$$

where  $h(x, t) = \text{div}A(x) - |A(x)|^2 + q(x, t)$ . Equating coefficients of power of  $|\sigma|$  to zero, we get  $\omega \cdot \nabla\phi(x) = -\omega \cdot A(x)$  for all  $x \in \mathbb{R}^n$ . Then  $w$  solves the following equation

$$(2.18) \quad (i\partial_t + \Delta_\rho + 2iA(x) \cdot \nabla_\rho + h(x, t)) w(x, t) = L(x, t),$$

where

$$(2.19) \quad L(x, t) = -e^{i\phi(x)} (i\Delta\phi(x) - |\nabla\phi(x)|^2 + 2y \cdot \nabla\phi(x) + 2A(x) \cdot y - 2A \cdot \nabla\phi(x) + h(x, t)).$$

In light of (2.18), we introduce the following map

$$\begin{aligned} U_\rho : L^2(0, T; H^1(\Omega)) &\longrightarrow L^2(0, T; H^1(\Omega)), \\ w &\longmapsto G_\rho(-w h + L). \end{aligned}$$

Applying (2.12) with  $k = 1$  and  $f = h(w - \tilde{w})$ , we get for all  $w, \tilde{w} \in L^2(0, T; H^1(\Omega))$  that

$$\begin{aligned} \|U_\rho(w) - U_\rho(\tilde{w})\|_{L^2(0, T; H^1(\Omega))} &= \|G_\rho(h(w - \tilde{w}))\|_{L^2(0, T; H^1(\Omega))} \\ &\leq \frac{C}{\sigma} \|h\|_{\mathcal{X}} \|w - \tilde{w}\|_{L^2(0, T; H^1(\Omega))}. \end{aligned}$$

Taking  $\sigma_0$  sufficiently large so that  $\sigma_0 > 2C\|h\|_{\mathcal{X}}$ , then, for each  $\sigma > \sigma_0$ ,  $U_\rho$  admits a unique fixed point  $w \in L^2(0, T; H^1(\Omega))$  such that  $U_\rho(w) = w$ . Again, applying (2.12) with  $k = 1$  and  $f = -hw + L$ , one gets

$$\begin{aligned} \|w\|_{L^2(0, T; H^1(\Omega))} &= \|G_\rho(-hw + L)\|_{L^2(0, T; H^1(\Omega))} \\ &\leq \frac{1}{2} \|w\|_{L^2(0, T; H^1(\Omega))} + \frac{C}{\sigma} \|L\|_{L^2(0, T; H^1(\Omega))}. \end{aligned}$$

Therefore, in view of Lemma 2.1 and (2.19), we get

$$(2.20) \quad \|w\|_{L^2(0, T; H^1(\Omega))} \leq \frac{C}{\sigma}.$$

Next, differentiating the equation (2.18) twice with respect to  $t$ , taking into account that  $\|h\|_{\mathcal{X}}$  is uniformly bounded with respect to  $\sigma$ , and proceeding as before, we show that

$$(2.21) \quad \|\partial_t^k w\|_{L^2(0, T; H^1(\Omega))} \leq \frac{C}{\sigma}, \quad k = 1, 2.$$

Finally, from (2.20) and Lemma 2.1, we obtain

$$\begin{aligned} \|w\|_{L^2(0, T; H^2(\Omega))} &\leq C\| -wh + L\|_{L^2(0, T; H^1(\Omega))} \\ &\leq C\left(\frac{C}{\sigma}\|h\|_{\mathcal{X}} + C\right) \\ (2.22) \quad &\leq C, \end{aligned}$$

by applying (2.12) with  $k = 2$  and  $f = -wh + L$ . Thus, we get the desired result by combining (2.20)-(2.22). □

### 3. STABILITY ESTIMATE FOR THE MAGNETIC FIELD

In this section, we prove Theorem 1.2 by means of the geometrical optics solutions

$$(2.23) \quad u_j(x, t) = e^{-i((\rho_j \cdot \rho_j)t + x \cdot \rho_j)} \left( e^{i\phi_j(x)} + w_j(x, t) \right), \quad j = 1, 2,$$

associated  $A_j$  and  $q_j$ . Here we choose  $\rho_j = \sigma\omega_j$  and we recall that the correction term  $w_j$  satisfies (2.17) and that  $\phi_j(x) = N_{\omega_j^*}^{-1}(-\omega_j^* \cdot A_j)$  solves the transport equation

$$\omega_j^* \cdot \nabla \phi_j(x) = -\omega_j^* \cdot A(x), \quad x \in \mathbb{R}^n.$$

Let us specify the choice of  $\rho_j$ : we consider  $\xi \in \mathbb{R}^n$  and  $\omega = \omega_{\Re} + i\omega_{\Im}$  with  $\omega_{\Re}, \omega_{\Im} \in \mathbb{S}^{n-1}$  and  $\omega_{\Re} \cdot \omega_{\Im} = \xi \cdot \omega_{\Re} = \xi \cdot \omega_{\Im} = 0$ . for each  $\sigma > |\xi|/2$ , we denote

$$(2.24) \quad \rho_1 = \sigma \left( i\omega_{\Im} + \left( -\frac{\xi}{2\sigma} + \sqrt{1 - \frac{|\xi|^2}{4\sigma^2}} \omega_{\Re} \right) \right) = \sigma\omega_1^*,$$

$$(2.25) \quad \rho_2 = \sigma \left( -i\omega_{\Im} + \left( \frac{\xi}{2\sigma} + \sqrt{1 - \frac{|\xi|^2}{4\sigma^2}} \omega_{\Re} \right) \right) = \sigma\omega_2^*.$$



Notice that  $\rho_j \cdot \rho_j = 0$ . In this section, we aim for recovering the magnetic field  $d\alpha_A$  from the boundary operator

$$\begin{aligned} \Lambda_{A,q} : L^2(\Omega) \times H^{2,1}(\Sigma) &\longrightarrow H^1(\Omega) \times L^2(\Sigma) \\ g = (u_0, f) &\longmapsto \left( u(\cdot, T), (\partial_\nu + iA \cdot \nu)u \right). \end{aligned}$$

We denote by

$$\Lambda_{A,q}^1 = u(\cdot, T), \quad \Lambda_{A,q}^2 = (\partial_\nu + iA \cdot \nu)u.$$

We first establish an orthogonality identity for the magnetic potential  $A = A_1 - A_2$ .

**3.1. A basic identity for the magnetic potential.** In this section, we derive an identity relating the magnetic potential  $A$  to the solutions  $u_j$ . We start by the following result.

**Lemma 3.1.** *Let  $\varepsilon > 0$ ,  $A_j \in \mathcal{A}_\varepsilon$  and  $u_j$  be the solutions given by (3.23)  $j = 1, 2$ . Then, for all  $\xi \in \mathbb{R}^n$  and  $\sigma > \max(\sigma_0, |\xi|/2)$ , we have*

$$\int_Q iA(x) \cdot (\overline{u_1} \nabla u_2 - u_2 \nabla \overline{u_1}) dx dt = \int_Q A(x) \cdot (\rho_2 + \overline{\rho_1}) e^{-ix \cdot \xi} e^{i(\phi_2 - \overline{\phi_1})(x)} + I(\xi, \sigma),$$

where the remaining term  $I(\xi, \sigma)$  is uniformly bounded with respect to  $\sigma$  and  $\xi$ .

*Proof.* In light of (3.23), we have by direct computation

$$\begin{aligned} \overline{u_1} \nabla u_2 - u_2 \nabla \overline{u_1} &= e^{-ix \cdot (\rho_2 - \overline{\rho_1})} \left[ -i\rho_2 e^{i(\phi_2 - \overline{\phi_1})} - i\overline{\rho_1} e^{i(\phi_2 - \overline{\phi_1})} \right. \\ &\quad + i\nabla \phi_2 e^{i(\phi_2 - \overline{\phi_1})} + i\nabla \overline{\phi_1} e^{i(\phi_2 - \overline{\phi_1})} - i\rho_2 w_2 e^{-i\overline{\phi_1}} - i\overline{\rho_1} w_1 e^{i\phi_2} \\ &\quad + \nabla w_2 e^{-i\overline{\phi_1}} - \nabla \overline{w_1} e^{i\phi_2} - i\rho_2 \overline{w_1} e^{i\phi_2} - i\overline{\rho_1} w_2 e^{-i\overline{\phi_1}} + i\nabla \phi_2 \overline{w_1} e^{i\phi_2} \\ &\quad \left. + iw_2 \nabla \overline{\phi_1} e^{-i\overline{\phi_1}} - i\rho_2 w_2 \overline{w_1} - i\overline{\rho_1} w_1 w_2 + \nabla w_2 \overline{w_1} - \nabla \overline{w_1} w_2 \right]. \end{aligned}$$

Therefore, as we have  $\rho_2 - \overline{\rho_1} = \xi$ , this yields that

$$\int_Q iA(x) \cdot (\overline{u_1} \nabla u_2 - u_2 \nabla \overline{u_1}) dx dt = \int_Q A(x) \cdot (\rho_2 + \overline{\rho_1}) e^{-ix \cdot \xi} e^{i(\phi_2 - \overline{\phi_1})} dx dt + I(\xi, \sigma),$$

where  $I(\xi, \sigma) = \int_Q iA(x) \cdot (\psi_1(x, t) + \psi_2(x, t)) dx dt$ , and  $\psi_1, \psi_2$  stand for

$$\psi_1 = -i(\rho_2 + \overline{\rho_1}) \left( w_2 e^{-i\overline{\phi_1}} + \overline{w_1} e^{i\phi_2} + w_2 \overline{w_1} \right),$$

$$\psi_2 = e^{i\phi_2} (i\nabla \phi_2 \overline{w_1} - \nabla \overline{w_1}) + e^{-i\overline{\phi_1}} (\nabla w_2 + i\nabla \overline{\phi_1} w_2) + \nabla w_2 \overline{w_1} - \nabla \overline{w_1} w_2 + i(\nabla \phi_2 + \nabla \overline{\phi_1}) e^{i(\phi_2 - \overline{\phi_1})}.$$

In view of bounding  $|I(\xi, \sigma)|$  uniformly with respect to  $\xi$  and  $\sigma$ , we use the fact that  $A$  is extended by zero outside  $\Omega$  and use Lemma 2.1 to get

$$\|\phi_j\|_{L^\infty(\Omega)} \leq C \|A_j\|_{L^\infty(\mathbb{R}^n)} \leq C\varepsilon, \quad j = 1, 2.$$

Recalling (2.16) and (2.17) and applying Lemma 2.1, we get

$$(3.26) \quad \|\psi_j\|_{L^1(Q)} \leq C \left( C + \frac{1}{\sigma} \right) \leq C, \quad j = 1, 2,$$

which yields the desired result.  $\square$

With the help of the above lemma we may now derive the following orthogonality identity for the magnetic potential.

**Lemma 3.2.** *Let  $\xi \in \mathbb{R}^n$  and  $\sigma > \max(\sigma_0, |\xi|/2)$ . Then, we have the following identity*

$$\int_Q A(x) \cdot (\rho_2 + \overline{\rho_1}) e^{-ix \cdot \xi} e^{i(\phi_2 - \overline{\phi_1})} dx dt = 2\sigma T \int_\Omega \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} dx + J(\xi, \sigma),$$

with  $|J(\xi, \sigma)| \leq C|\xi|$ , where  $C$  is independent of  $\sigma$  and  $\xi$ .

*Proof.* In view of (3.24) and (3.25), we have

$$(3.27) \quad \begin{aligned} \int_Q A(x) \cdot (\rho_2 + \overline{\rho_1}) e^{-ix \cdot \xi} e^{i(\phi_2 - \overline{\phi_1})} dx dt &= 2\sigma \int_Q \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} e^{i(\phi_2 - \overline{\phi_1})} dx dt \\ &\quad - 2\sigma \left(1 - \sqrt{1 - |\xi|^2/4\sigma^2}\right) \int_Q \omega_{\mathbb{R}} \cdot A(x) e^{-ix \cdot \xi} e^{i(\phi_2 - \overline{\phi_1})} dx dt, \end{aligned}$$

where we recall that

$$\overline{\phi_1} = N_{\omega_1^*}^{-1}(-\overline{\omega_1^*} \cdot A_1), \quad \phi_2 = N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_2).$$

Set  $\overline{\Psi_1} = N_{\overline{\omega}}^{-1}(-\overline{\omega} \cdot A_1)$  and  $\Psi_2 = N_{\overline{\omega}}^{-1}(-\overline{\omega} \cdot A_2)$  in such away that we have

$$\Psi_2 - \overline{\Psi_1} = N_{\overline{\omega}}^{-1}(-(-\overline{\omega} \cdot A)) = -N_{\overline{\omega}}^{-1}(-\overline{\omega} \cdot A).$$

Then, we infer from (3.27) that

$$\int_Q A(x) \cdot (\rho_2 + \overline{\rho_1}) e^{-ix \cdot \xi} e^{i(\phi_2 - \overline{\phi_1})} dx dt = J_1(\xi, \sigma) + J_2(\xi, \sigma) + J_3(\xi, \sigma),$$

where we have set

$$\begin{aligned} J_1(\xi, \sigma) &= 2\sigma \int_Q \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} e^{i(\Psi_2 - \overline{\Psi_1})} dx dt, \\ J_2(\xi, \sigma) &= -2\sigma \int_Q \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} \left( e^{i(\Psi_2 - \overline{\Psi_1})} - e^{i(\phi_2 - \overline{\phi_1})} \right) dx dt, \end{aligned}$$

and

$$J_3(\xi, \sigma) = -2\sigma \left(1 - \sqrt{1 - |\xi|^2/4\sigma^2}\right) \int_Q \omega_{\mathbb{R}} \cdot A(x) e^{-ix \cdot \xi} e^{i(\phi_2 - \overline{\phi_1})} dx dt.$$

Using Lemma 2.2, one can see that

$$\begin{aligned} J_1(\xi, \sigma) &= 2\sigma T \int_\Omega \overline{\omega} \cdot A(x) e^{iN_{\overline{\omega}}^{-1}(-\overline{\omega} \cdot (-A))} e^{-ix \cdot \xi} dx \\ &= 2\sigma T \int_\Omega \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} dx dt. \end{aligned}$$

Now it remains to upper bound the absolute value of  $J := J_2 + J_3$ . We start by inserting  $e^{i(\Psi_2 - \overline{\phi_1})}$  into  $J_2(\xi, \sigma)$ , getting

$$J_2(\xi, \sigma) = -2\sigma T \int_\Omega \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} \left( e^{i\Psi_2} \left( e^{-i\overline{\Psi_1}} - e^{-i\overline{\phi_1}} \right) + e^{-i\overline{\phi_1}} \left( e^{i\Psi_2} - e^{i\phi_2} \right) \right) dx.$$

Further, as  $N_{\omega}^{-1}(-\omega \cdot A)$  depends continuously on  $\omega$ , according to Lemma 2.4 in [25], we get for all  $|\xi| \leq 2\sigma$

$$|J_2(\xi, \sigma)| \leq C_T \sigma (|\overline{\omega} - \overline{\omega_1^*}| + |\overline{\omega} - \omega_2^*|).$$

Hence, as  $1 - \sqrt{1 - |\xi|^2/4\sigma^2} \leq |\xi|^2/4\sigma^2$  for all  $|\xi| \leq 2\sigma$ , we deduce from (3.24), (3.25) and the above inequality that

$$|J_2(\xi, \sigma)| \leq C_T \left( \sigma \frac{|\xi|^2}{4\sigma^2} + |\xi| \right) \leq C_T |\xi|.$$

Arguing in the same way, we find that  $|J_3(\xi, \sigma)| \leq C_T |\xi|$ , for some positive constant  $C_T$  which is independent of  $\xi$  and  $\sigma$ .  $\square$

**3.2. Estimating the Fourier transform of the magnetic field.** We aim to relate the Fourier transform of the magnetic field  $d\alpha_{A_1} - d\alpha_{A_2}$  to the measurement  $\Lambda_{A_1, q_1} - \Lambda_{A_2, q_2}$ . To this end, we introduce the following notation: we put

$$a_k(x) = (A_1 - A_2)(x) \cdot e_k = A(x) \cdot e_k,$$

where  $(e_k)_k$  is the canonical basis of  $\mathbb{R}^n$ , and

$$(3.28) \quad \sigma_{j,k}(x) = \frac{\partial a_k}{\partial x_j}(x) - \frac{\partial a_j}{\partial x_k}(x), \quad j, k = 1, \dots, n.$$

We recall that the Green formula for the magnetic Laplacian

$$(3.29) \quad \int_{\Omega} (\Delta_A u \bar{v} - u \overline{\Delta_A v}) dx = - \int_{\Gamma} ((\partial_{\nu} + i\nu \cdot A)u \bar{v} - u \overline{(\partial_{\nu} + iA \cdot \nu)v}) d\sigma_x,$$

holds for any  $u, v \in H^1(\Omega)$  such that  $\Delta u, \Delta v \in L^2(\Omega)$ . Here  $d\sigma_x$  is the Euclidean surface measure on  $\Gamma$ . We estimate the Fourier transform of  $\sigma_{j,k}$  as follows.

**Lemma 3.3.** *Let  $\xi \in \mathbb{R}^n$  and  $\sigma > \max(\sigma_0, |\xi|/2)$ , where  $\sigma_0$  is as in Lemma 2.7. Then we have*

$$< \xi >^{-1} |\hat{\sigma}_{j,k}(\xi)| \leq C \left( e^{C\sigma} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{|\sigma|} \right),$$

where  $C$  is independent of  $\xi$  and  $\sigma$ .

*Proof.* First, for  $\sigma > \sigma_0$ , Lemma 2.7 guarantees the existence of a geometrical optic solution  $u_2$ , of the form

$$u_2(x, t) = e^{-ix \cdot \rho_2} (e^{i\phi_2(x)} + w_2(x, t))$$

to the magnetic Schrödinger equation

$$(3.30) \quad \begin{cases} (i\partial_t + \Delta_{A_2} + q_2(x, t))u_2(x, t) = 0, & \text{in } Q, \\ u_2(x, 0) = u_0, & \text{in } \Omega, \end{cases}$$

where  $\rho_2$  is given by (3.25). Let us denote by  $f_{\sigma} := u_2|_{\Sigma}$ . We consider a solution  $v$  to the following non homogeneous boundary value problem

$$(3.31) \quad \begin{cases} (i\partial_t + \Delta_{A_1} + q_1(x, t))v = 0, & \text{in } Q, \\ v(., 0) = u_2(., 0) = u_0, & \text{in } \Omega, \\ v = u_2 = f_{\sigma}, & \text{on } \Sigma. \end{cases}$$

Then,  $u = v - u_2$  is a solution to the following homogenous boundary value problem for the magnetic Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_{A_1} + q_1(x, t))u = 2iA \cdot \nabla u_2 + h(x, t)u_2, & \text{in } Q, \\ u(x, 0) = 0, & \text{in } \Omega, \\ u(x, t) = 0, & \text{on } \Sigma, \end{cases}$$

where

$$A = A_1 - A_2, \quad q = q_1 - q_2 \quad \text{and} \quad h = i \operatorname{div} A - (|A_1|^2 - |A_2|^2) + q.$$

On the other hand, with reference to Lemma 2.7 we consider a solution  $u_1$  to the magnetic Schrödinger equation (2.14), associated with the potentials  $A_1$  and  $q_1$ , of the form

$$u_1(x, t) = e^{-ix \cdot \rho_1} (e^{i\phi_1(x)} + w_1(x, t)),$$

where  $\rho_1$  is given by (3.24). Integrating by parts in the following integral, and using the Green Formula (3.29), we get

$$(3.32) \quad \begin{aligned} \int_Q (i\partial_t + \Delta_{A_1} + q_1) u \overline{u_1} dx dt &= \int_Q 2iA \cdot \nabla u_2 \overline{u_1} dx dt + \int_Q \left( i \operatorname{div} A - (|A_1|^2 - |A_2|^2) + q \right) u_2 \overline{u_1} dx dt \\ &= i \int_{\Omega} u(\cdot, T) \overline{u_1}(\cdot, T) dx - \int_{\Sigma} (\partial_\nu + iA_1 \cdot \nu) u \overline{u_1} d\sigma_x dt. \end{aligned}$$

This entails that

$$\begin{aligned} \int_Q 2iA \cdot \nabla u_2 \overline{u_1} dx dt &= -i \int_{\Omega} (\Lambda_{A_2, q_2}^1 - \Lambda_{A_1, q_1}^1)(g) \overline{u_1}(\cdot, T) dx + \int_{\Sigma} (\Lambda_{A_2, q_2}^2 - \Lambda_{A_1, q_1}^2)(g) \overline{u_1} d\sigma_x dt \\ &\quad - \int_Q \left( i \operatorname{div} A - (|A_1|^2 - |A_2|^2) + q \right) u_2 \overline{u_1} dx dt, \end{aligned}$$

where  $g = (u_2|_{t=0}, u_2|_{\Sigma})$ . Upon applying the Stokes formula and using the fact that  $A|_{\Gamma} = 0$ , we get

$$(3.33) \quad \begin{aligned} \int_Q iA \cdot (\overline{u_1} \nabla u_2 - u_2 \nabla \overline{u_1}) dx dt &= -i \int_{\Omega} (\Lambda_{A_2, q_2}^1 - \Lambda_{A_1, q_1}^1)(g) \overline{u_1}(\cdot, T) dx + \int_{\Sigma} (\Lambda_{A_2, q_2}^2 - \Lambda_{A_1, q_1}^2)(g) \overline{u_1} d\sigma_x dt \\ &\quad + \int_Q (|A_1|^2 - |A_2|^2 + q) u_2 \overline{u_1} dx dt. \end{aligned}$$

This, Lemma 3.1 and Lemma 3.2, yield

$$\left| \int_{\Omega} \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} dx \right| \leq \frac{C_T}{\sigma} \left( \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| \|g\|_{H^2(\Omega) \times H^{2,1}(\Sigma)} \|\phi\|_{L^2(\Sigma) \times L^2(\Omega)} + C + |\xi| \right),$$

where  $\phi = (\overline{u_1}|_{\Sigma}, \overline{u_1}|_{t=T})$ . Here we used the fact that  $\|u_2 \overline{u_1}\|_{L^1(Q)} \leq C_T$ , for  $\sigma$  sufficiently large. Hence, bearing in mind that

$$\|g\|_{H^2(\Omega) \times H^{2,1}(\Sigma)} \leq C e^{C\sigma}, \quad \text{and} \quad \|\phi\|_{L^2(\Sigma) \times L^2(\Omega)} \leq C e^{C\sigma},$$

we get for  $\sigma > |\xi|/2$ ,

$$(3.34) \quad \left| \int_{\Omega} \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} dx \right| \leq C \left( e^{C\sigma} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{\sigma} \right).$$

Arguing as in the derivation of (3.34), we prove by replacing  $\overline{\omega}$  by  $-\omega$ , that

$$(3.35) \quad \left| \int_{\Omega} -\omega \cdot A(x) e^{-ix \cdot \xi} dx \right| \leq C \left( e^{C\sigma} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{\sigma} \right).$$

Thus, choosing  $\omega_{\mathfrak{S}} = \frac{\xi_j e_k - \xi_k e_j}{|\xi_j e_k - \xi_k e_j|}$ , multiplying (3.34) and (3.35) by  $|\xi_j e_k - \xi_k e_j|$ , and adding the obtained inequalities together, we find that

$$\left| \int_{\Omega} e^{-ix \cdot \xi} (\xi_j \tilde{a}_k(x) - \xi_k \tilde{a}_j(x)) dx \right| \leq C |\xi_j e_k - \xi_k e_j| \left( e^{C\sigma} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{\sigma} \right).$$

From this and (3.28) we deduce that

$$|\hat{\sigma}_{j,k}(\xi)| \leq C < \xi > \left( e^{C\sigma} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{\sigma} \right), \quad j, k \in \mathbb{N}.$$

This ends the proof.  $\square$

**3.3. Stability estimate.** Armed with Lemma 3.3, we are now in position to complete the proof of the stability estimate for the magnetic field. To do so, we first need to bound the  $H^{-1}(\mathbb{R}^n)$  norm of  $d\alpha_{A_1} - d\alpha_{A_2}$ . In light of the above reasoning, this can be achieved by taking  $\sigma > R > 0$  and decomposing the  $H^{-1}(\mathbb{R}^n)$  norm of  $\sigma_{j,k}$  as

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)}^2 = \int_{|\xi| \leq R} |\hat{\sigma}_{j,k}(\xi)|^2 \langle \xi \rangle^{-2} d\xi + \int_{|\xi| > R} |\hat{\sigma}_{j,k}(\xi)|^2 \langle \xi \rangle^{-2} d\xi.$$

Then, we have

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)}^2 \leq C \left[ R^n \|\langle \xi \rangle^{-1} \hat{\sigma}_{j,k}\|_{L^\infty(B(0,R))}^2 + \frac{1}{R^2} \|\sigma_{j,k}\|_{L^2(\mathbb{R}^n)}^2 \right],$$

which entails that

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)}^2 \leq C \left[ R^n \left( e^{C\sigma} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^2 + \frac{1}{\sigma^2} + \frac{R^2}{\sigma^2} \right) + \frac{1}{R^2} \right],$$

by Lemma 3.3. The next step is to choose  $R > 0$  in such away  $\frac{R^{n+2}}{\sigma^2} = \frac{1}{R^2}$ . In this case we get for  $\sigma > \max(\sigma_0, |\xi|/2)$ , that

$$\begin{aligned} \|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)}^2 &\leq C \left( \sigma^{\frac{2n}{n+4}} e^{C\sigma} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^2 + \sigma^{\frac{-4}{n+4}} \right) \\ (3.36) \quad &\leq C \left( e^{C_0\sigma} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^2 + \frac{1}{\sigma^\mu} \right), \end{aligned}$$

where  $\mu \in (0, 1)$ . Thus, assuming that  $\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| \leq c = e^{-C_0 \max(\sigma_0, |\xi|/2)}$ , and taking  $\sigma = \frac{1}{C_0} |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||$  in (3.36), we get that

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)} \leq C \left( \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu'} \right),$$

for some positive  $\mu' \in (0, 1)$ . Since the above estimate remains true when  $\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| \geq c$ , as we have

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)} \leq \frac{2M}{c^{1/2}} c^{1/2} \leq \frac{2M}{c^{1/2}} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2},$$

we have obtained that

$$\|d\alpha_{A_1} - d\alpha_{A_2}\|_{H^{-1}(\Omega)} \leq C \left( \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu'} \right).$$

In order to complete the proof of the theorem, we consider  $\delta > 0$  such that  $\alpha := s - 1 = \frac{n}{2} + 2\delta$ , use Sobolev's embedding theorem and we find

$$\begin{aligned} \|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} &\leq C \|d\alpha_{A_1} - d\alpha_{A_2}\|_{H^{\frac{n}{2}+\delta}(\Omega)} \\ &\leq C \|d\alpha_{A_1} - d\alpha_{A_2}\|_{H^{-1}(\Omega)}^{1-\beta} \|d\alpha_{A_1} - d\alpha_{A_2}\|_{H^{s-1}(\Omega)}^\beta \\ &\leq C \left( \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu} \right)^{1-\beta}, \end{aligned}$$

by interpolating with  $\beta \in (0, 1)$ . This completes the proof of Theorem 1.2.

This theorem is a key ingredient in the proof of the result of the next section.

#### 4. STABILITY RESULT FOR THE ELECTRIC POTENTIAL

This section contains the proof of Theorem 1.3. Using the geometric optics solutions constructed in Section 2, we will prove with the aid of the stability estimate obtained for the magnetic field, that the time-dependent electric potential depends stably on the Dirichlet-to-Neumann map  $\Lambda_{A,q}$ .

To do this, we should normally apply the Hodge decomposition to  $A = A_1 - A_2 = A' + \nabla\varphi$  and use this estimate

$$(4.37) \quad \|A'\|_{W^{1,p}(\Omega)} \leq C \|\operatorname{curl} A'\|_{L^p(\Omega)}.$$

that holds for any  $p > n$  (see Appendix B). But in this paper, since  $u_0$  is not frozen to zero, we don't have invariance under Gauge transformation, so will further assume that  $A$  is divergence free in such a way that the estimate (4.37) holds for  $A' = A$ .

For a fixed  $y \in B(0, 1)$ , we consider solutions  $u_j$  to the Schrödinger equation of the form (3.23) with  $\rho_j = \sigma\omega_j^* + y$ , where  $\xi \in \mathbb{R}^n$  and  $\omega \in \mathbb{S}^{n-1}$  are as in Section 3, and  $w_j^*$ ,  $j = 1, 2$ , are given by (3.24) and (3.25).

In contrast to Section 3,  $y$  is no longer equal to zero, as we need to estimate the Fourier transform of  $q$  with respect to  $x$  and  $t$ .

**4.1. An identity for the electric potential.** Let us first establish the following identity for the electric potential.

**Lemma 4.1.** *Let  $u_j$  be the solutions given by (3.23) for  $j = 1, 2$ . For all  $\sigma \geq \sigma_0$  and  $\xi \in \mathbb{R}^n$  such that  $|\xi| < 2\sigma$ , we have the following identity*

$$\int_Q q(x, t) u_2 \overline{u_1} dx dt = \int_Q q(x, t) e^{-i(2y \cdot \xi t + x \cdot \xi)} dx dt + P_1(\xi, y, \sigma) + P_2(\xi, y, \sigma),$$

where  $P_1(\xi, y, \sigma)$  and  $P_2(\xi, y, \sigma)$  satisfy the estimates

$$|P_1(\xi, y, \sigma)| \leq C \left( \|A\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} \right), \quad |P_2(\xi, y, \sigma)| \leq \frac{C}{\sigma}.$$

Here  $\sigma_0$  is as in Lemma 2.7 and  $C$  is independent of  $\sigma$ ,  $y$ , and  $\xi$ .

*Proof.* In light of (3.24), (3.25) and (3.23), a direct calculation gives us

$$(4.38) \quad \begin{aligned} u_2 \overline{u_1} &= e^{-i \left( (\rho_2 \cdot \rho_2 - \overline{\rho_1} \cdot \rho_1) t + x \cdot (\rho_2 - \overline{\rho_1}) \right)} \left( e^{i(\phi_2 - \overline{\phi_1})} + e^{-i\overline{\phi_1}} w_2 + e^{i\phi_2} \overline{w_1} + w_2 \overline{w_1} \right) \\ &= e^{-i(2y \cdot \xi t + x \cdot \xi)} e^{-i(\overline{\phi_1} - \phi_2)} + e^{-i(2y \cdot \xi t + x \cdot \xi)} \left( e^{-i\overline{\phi_1}} w_2 + e^{i\phi_2} \overline{w_1} + w_2 \overline{w_1} \right), \end{aligned}$$

which yields

$$(4.39) \quad \int_Q q(x, t) u_2 \overline{u_1} dx dt = \int_Q q(x, t) e^{-i(2y \cdot \xi t + x \cdot \xi)} dx dt + P_1(\xi, y, \sigma) + P_2(\xi, y, \sigma),$$

where we have set

$$\begin{aligned} P_1(\xi, y, \sigma) &= \int_Q q(x, t) e^{-i(2y \cdot \xi t + x \cdot \xi)} e^{-i\overline{\phi_1}} \left( e^{i\phi_2} - e^{i\overline{\phi_1}} \right) dx dt, \\ P_2(\xi, y, \sigma) &= \int_Q q(x, t) e^{-i(2y \cdot \xi t + x \cdot \xi)} \left( e^{-i\overline{\phi_1}} w_2 + e^{i\phi_2} \overline{w_1} + w_2 \overline{w_1} \right) dx dt. \end{aligned}$$

Recalling that  $\phi_j = N_{\omega_j^*}^{-1}(-\omega_j^* \cdot A_j)$ , for  $j = 1, 2$ , we deduce from the definition of  $P_1$  that

$$|P_1(\xi, y, \sigma)| \leq C \left( \|e^{iN_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_2)} - e^{iN_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_1)}\|_{L^\infty(\Omega)} + \|e^{iN_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_1)} - e^{iN_{\bar{\omega}_1^*}^{-1}(-\bar{\omega}_1^* \cdot A_1)}\|_{L^\infty(\Omega)} \right),$$

with  $C > 0$  is depending on  $T, M, \Omega$  and  $\|A_1\|$ . Using the continuity of  $N_\omega^{-1}(-\omega \cdot A)$  with respect to  $\omega$  (see Lemma 2.4 in [25]), we get that

$$\begin{aligned} |P_1(\xi, y, \sigma)| &\leq C \left( \|N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_2) - N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_1)\|_{L^\infty(\Omega)} + |\omega_2^* - \bar{\omega}_1^*| \right) \\ &\leq C \left( \|A\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} \right). \end{aligned}$$

On the other hand, from Cauchy Schwarz inequality, Lemma 2.1 and (2.17), we get

$$\begin{aligned} |P_2(\xi, y, \sigma)| &\leq C \left( \|w_2\|_{L^2(Q)} \|e^{-i\bar{\phi}_1}\|_{L^2(Q)} + \|e^{i\phi_2}\|_{L^2(Q)} \|\bar{w}_1\|_{L^2(Q)} + \|w_2\|_{L^2(Q)} \|\bar{w}_1\|_{L^2(\Omega)} \right) \\ &\leq \frac{C}{\sigma}. \end{aligned}$$

This completes the proof of Lemma 4.1.  $\square$

**4.2. Estimate of the Fourier transform.** In view of relating the Fourier transform of the electric potential  $q = q_1 - q_2$  to  $\Lambda_{A_1, q_1} - \Lambda_{A_2, q_2}$ , we first establish the following auxiliary result

**Lemma 4.2.** *For any  $\sigma \geq \sigma_0$  and  $\xi \in \mathbb{R}^n$  such that  $|\xi| < 2\sigma$ , we have the following estimate*

$$|\hat{q}(\xi, 2y \cdot \xi)| \leq C \left( e^{C\sigma} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| + e^{C\sigma} \|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} + \frac{1}{\sigma} \right),$$

for some  $C$  that is independent of  $|\xi|$  and  $\sigma$ .

*Proof.* First, for  $\sigma > \sigma_0$ , Lemma 2.7 guarantees the existence of a geometrical optics solution  $u_2$  of the form

$$u_2(x, t) = e^{-i((\rho_2 \cdot \rho_2)t + x \cdot \rho_2)} (e^{i\phi_2(x)} + w_2(x, t)),$$

to the magnetic Schrödinger equation

$$(4.40) \quad \begin{cases} (i\partial_t + \Delta_{A_2} + q_2(x, t))u_2(x, t) = 0, & \text{in } Q, \\ u_2(x, 0) = u_0, & \text{in } \Omega, \end{cases}$$

where  $\rho_2$  is given by (3.25) and  $w_2(x, t)$  satisfies

$$(4.41) \quad \sigma \|w_2\|_{H^2(0, T, H^1(\Omega))} + \|w_2\|_{L^2(0, T, H^2(\Omega))} \leq C$$

Let us denote by  $f_\sigma := u_2|_\Sigma$ . We consider a solution  $v$  to the following non homogeneous boundary value problem

$$(4.42) \quad \begin{cases} (i\partial_t + \Delta_{A_1} + q_1(x, t))v = 0, & \text{in } Q, \\ v(\cdot, 0) = u_2(\cdot, 0) = u_0, & \text{in } \Omega, \\ v = u_2 = f_\sigma, & \text{on } \Sigma. \end{cases}$$

Denote  $u = v - u_2$ , then  $u$  is a solution to the following homogenous boundary value problem for the magnetic Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_{A_1} + q_1(x, t))u = 2iA \cdot \nabla u_2 + h(x, t)u_2, & \text{in } Q, \\ u(x, 0) = 0, & \text{in } \Omega, \\ u(x, t) = 0, & \text{on } \Sigma, \end{cases}$$

where we recall that

$$A = A_1 - A_2, \quad q = q_1 - q_2 \quad \text{and} \quad h = i \operatorname{div} A - (|A_1|^2 - |A_2|^2) + q.$$

On the other hand, we consider a solution  $u_1$  of the magnetic Shrödinger equation (2.14) corresponding to the potentials  $A_1$  and  $q_1$ , of the form

$$u_1(x, t) = e^{-i((\rho_1 \cdot \rho_1)t + x \cdot \rho_1)} (e^{i\phi_1(x)} + w_1(x, t)),$$

where  $\rho_1$  is given by (3.24) and  $w_1(x, t)$  satisfies

$$(4.43) \quad \sigma \|w_1\|_{H^2(0,T,H^1(\Omega))} + \|w_1\|_{L^2(0,T,H^2(\Omega))} \leq C.$$

Integrating by parts and using the Green Formula (3.29), we get

$$\begin{aligned} \int_Q q(x, t) u_2 \overline{u_1} dx dt &= i \int_{\Omega} (\Lambda_{A_2, q_2}^1 - \Lambda_{A_1, q_1}^1)(g) \overline{u_1}(\cdot, T) dx - \int_{\Sigma} (\Lambda_{A_2, q_2}^1 - \Lambda_{A_1, q_1}^2)(g) \overline{u_1} d\sigma_x dt \\ &\quad + \int_Q iA(x) \cdot (\overline{u_1} \nabla u_2 - u_2 \nabla \overline{u_1}) dx dt - \int_Q (|A_1|^2 - |A_2|^2) u_2 \overline{u_1} dx dt, \end{aligned}$$

where  $g = (u_2|_{t=0}, u_2|_{\Sigma})$ . To bring the Fourier transform of  $q$  out of the above identity, we extend  $q$  by zero outside the cylindrical domain  $Q$ , we use Lemma 4.1 and take to account that

$$\|u_2 \overline{u_1}\|_{L^1(Q)} \leq C, \quad \text{and} \quad \|\overline{u_1} \nabla u_2\|_{L^1(Q)} + \|u_2 \nabla \overline{u_1}\|_{L^1(Q)} \leq C\sigma,$$

and get

$$|\hat{q}(\xi, 2y \cdot \xi)| \leq C \left( \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| \|g\|_{H^2(\Omega) \times H^{2,1}(\Sigma)} \|\phi\|_{L^2(\Sigma) \times L^2(\Omega)} + C\sigma \|A\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} + \frac{1}{\sigma} \right),$$

where  $\phi = (\overline{u_1}|_{\Sigma}, \overline{u_1}|_{t=T})$ . Now, bearing in mind that

$$\|g\|_{H^2(\Omega) \times H^{2,1}(\Sigma)} \leq Ce^{C\sigma}, \quad \text{and} \quad \|\phi\|_{L^2(\Sigma) \times L^2(\Omega)} \leq Ce^{C\sigma},$$

we get for all  $\xi \in \mathbb{R}^n$  such that  $|\xi| < 2\sigma$  and for all  $y \in B(0, 1)$ ,

$$(4.44) \quad |\hat{q}(\xi, 2y \cdot \xi)| \leq C \left( e^{C\sigma} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| + e^{C\sigma} \|A\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} + \frac{1}{\sigma} \right).$$

Finally, using the fact that  $\|A\|_{W^{1,\infty}(\Omega)} \leq C \|\operatorname{curl} A\|_{L^\infty(\Omega)}$ , (see Lemma B.5 in Appendix B), we obtain the desired result.  $\square$

We are now in position to estimate  $\hat{q}(\xi, \tau)$  for all  $(\xi, \tau)$  in the following set

$$E_\alpha = \{(\xi, \tau) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}, \quad |\xi| < 2\alpha, \quad |\tau| < 2|\xi|\},$$

for any fixed  $0 < \alpha < \sigma$ .

**Lemma 4.3.** *Suppose that the conditions of Lemma 4.2 are satisfied. Then we have for all  $(\xi, \tau) \in E_\alpha$ ,*

$$(4.45) \quad |\hat{q}(\xi, \tau)| \leq C \left( e^{C\sigma} \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\| + e^{C\sigma} \|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} + \frac{\alpha}{\sigma} + \frac{1}{\sigma} \right).$$

Here  $C$  is independent of  $|\xi|$  and  $\sigma$ .

*Proof.* Fix  $(\xi, \tau) \in E_\alpha$ , and set  $y = \frac{\tau}{2|\xi|^2} \cdot \xi$ , in such away that  $y \in B(0, 1)$  and  $2y \cdot \xi = \tau$ . Since  $\alpha < \sigma$  we have  $|\xi| < 2\alpha < 2\sigma$ . Hence, Lemma 4.2 yields the desired result.  $\square$



**4.3. Stability estimate.** In order to complete the proof of the stability estimate for the electric potential, we use an argument for analytic functions proved in [3] (see also [1, 26]). For  $\gamma \in \mathbb{N}^{n+1}$ , we put  $|\gamma| = \gamma_1 + \dots + \gamma_{n+1}$ . We have the following statement that claims conditional stability for the analytic continuation.

**Lemma 4.4.** *Let  $O$  be a non empty open set of  $B(0, 1)$  and let  $F$  be an analytic function in  $B(0, 2)$ , obeying*

$$\|\partial^\gamma F\|_{L^\infty(B(0,2))} \leq \frac{M|\gamma|}{\eta^{|\gamma|}}, \quad \forall \gamma \in \mathbb{N}^{n+1}$$

for some  $M > 0$  and  $\eta > 0$ . Then we have

$$\|F\|_{L^\infty(B(0,1))} \leq (2M)^{1-\mu} \|F\|_{L^\infty(O)}^\mu,$$

where  $\mu \in (0, 1)$  depends on  $n$ ,  $\eta$  and  $|O|$ .

We refer to Lavrent'ev [12] for classical results for this type. For fixed  $0 < \alpha < \sigma$ , let us set

$$F_\alpha(\xi, \tau) = \widehat{q}(\alpha(\xi, \tau)), \quad (\xi, \tau) \in \mathbb{R}^{n+1}.$$

It is easily seen that  $F_\alpha$  is analytic and that

$$\begin{aligned} |\partial^\gamma F_\alpha(\xi, \tau)| &= |\partial^\gamma \widehat{q}(\alpha(\xi, \tau))| = \left| \partial^\gamma \int_{\mathbb{R}^{n+1}} q(x, t) e^{-\alpha(x, t) \cdot (\tau, \xi)} dx dt \right| \\ &= \left| \int_{\mathbb{R}^{n+1}} q(x, t) (-i)^{|\gamma|} \alpha^{|\gamma|}(x, t)^\gamma e^{-i\alpha(x, t) \cdot (\xi, \tau)} dx dt \right|. \end{aligned}$$

Hence one gets

$$|\partial^\gamma F_\alpha(\xi, \tau)| \leq \int_{\mathbb{R}^{n+1}} |q(x, t)| \alpha^{|\gamma|} (|x|^2 + t^2)^{\frac{|\gamma|}{2}} dx dt \leq \|q\|_{L^1(Q)} \alpha^{|\gamma|} (2T^2)^{\frac{|\gamma|}{2}} \leq C \frac{|\gamma|!}{(T^{-1})^{|\gamma|}} e^\alpha.$$

Applying Lemma 4.4 on the set  $O = E_1 \cap B(0, 1)$  with  $M = Ce^\alpha$ ,  $\eta = T^{-1}$ , we may find a constant  $\mu \in (0, 1)$  such that we have

$$|F_\alpha(\xi, \tau)| = |\widehat{q}(\alpha(\xi, \tau))| \leq Ce^{\alpha(1-\mu)} \|F_\alpha\|_{L^\infty(O)}^\mu, \quad (\xi, \tau) \in B(0, 1).$$

Now the idea is to estimate the Fourier transform of  $q$  in a suitable ball. Bearing in mind that  $\alpha E_1 = E_\alpha$ , we have for all  $(\xi, \tau) \in B(0, \alpha)$ ,

$$\begin{aligned} |\widehat{q}(\xi, \tau)| &= |F_\alpha(\alpha^{-1}(\xi, \tau))| \leq Ce^{\alpha(1-\mu)} \|F_\alpha\|_{L^\infty(O)}^\mu \\ &\leq Ce^{\alpha(1-\mu)} \|\widehat{q}\|_{L^\infty(B(0, \alpha) \cap E_\alpha)}^\mu \\ (4.46) \quad &\leq Ce^{\alpha(1-\mu)} \|\widehat{q}\|_{L^\infty(E_\alpha)}^\mu. \end{aligned}$$

The next step of the proof is to get an estimate linking the coefficient  $q$  to the measurement  $\Lambda_{A_1, q_1} - \Lambda_{A_2, q_2}$ . To do that we first decompose the  $H^{-1}(\mathbb{R}^{n+1})$  norm of  $q$  as follows

$$\begin{aligned} \|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\mu}} &= \left( \int_{|(\xi, \tau)| < \alpha} \langle (\xi, \tau) \rangle^{-2} |\widehat{q}(\xi, \tau)|^2 d\xi d\tau + \int_{|(\xi, \tau)| \geq \alpha} \langle (\xi, \tau) \rangle^{-2} |\widehat{q}(\xi, \tau)|^2 d\xi d\tau \right)^{\frac{1}{\mu}} \\ &\leq C \left( \alpha^{n+1} \|\widehat{q}\|_{L^\infty(B(0, \alpha))}^2 + \alpha^{-2} \|q\|_{L^2(\mathbb{R}^{n+1})}^2 \right)^{\frac{1}{\mu}}. \end{aligned}$$

It follows from (4.46) and Lemma 4.3, that

$$(4.47) \quad \|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\mu}} \leq C \left[ \alpha^{\frac{n+1}{\mu}} e^{\frac{2\alpha(1-\mu)}{\mu}} \left( e^{C\sigma} \eta^2 + e^{C\sigma} \|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)}^2 + \frac{\alpha^2}{\sigma^2} + \frac{1}{\sigma^2} \right) + \frac{1}{\alpha^\mu} \right],$$

where we have set  $\eta = \|\Lambda_{A_2, q_2} - \Lambda_{A_1, q_1}\|$ . In light of Theorem 1.2, one gets

$$(4.48) \quad \|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\mu}} \leq C \left[ \alpha^{\frac{n+1}{\mu}} e^{\frac{2\alpha(1-\mu)}{\mu}} \left( e^{C\sigma} \eta^2 + e^{C\sigma} \eta^s + e^{C\sigma} |\log \eta|^{-2\mu s} + \frac{\alpha^2}{\sigma^2} + \frac{1}{\sigma^2} \right) + \frac{1}{\alpha^{\frac{2}{\mu}}} \right].$$

The above statements are valid provided  $\sigma$  is sufficiently large. Then, we choose  $\alpha$  so large that  $\sigma = \alpha^{\frac{2\mu+n+3}{2\mu}} e^{\frac{\alpha(1-\mu)}{\mu}}$ , and hence  $\alpha^{\frac{2\mu+n+1}{\mu}} e^{\frac{2\alpha(1-\mu)}{\mu}} \sigma^{-2} = \alpha^{\frac{-2}{\mu}}$ , so the estimate (4.48) yields

$$(4.49) \quad \|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\mu}} \leq C \left[ e^{Ce^{N\alpha}} (\eta^2 + \eta^s + |\log \eta|^{-2\mu s}) + \alpha^{\frac{-2}{\mu}} \right],$$

where  $N$  depends on  $\mu$  and  $n$ . Thus, if  $\eta \in (0, 1)$ , we have

$$(4.50) \quad \|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\mu}} \leq C \left( e^{Ce^{N\alpha}} |\log \eta|^{-2\mu s} + \alpha^{\frac{-2}{\mu}} \right).$$

Finally, if  $\eta$  is small enough, taking  $\alpha = \frac{1}{N} \log \left( \log |\log \eta|^{\frac{\mu s}{c}} \right)$ , we get from (4.50) that

$$\|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\mu}} \leq C \left[ |\log \eta|^{-\mu s} + \left[ \log \left( \log |\log \eta|^{\frac{\mu s}{c}} \right) \right]^{-\frac{2}{\mu}} \right].$$

This completes the proof of Theorem 1.3.

## APPENDIX A. WELL-POSEDNESS OF THE MAGNETIC SCHRÖDINGER EQUATION

In this section we will establish the existence, uniqueness and continuous dependence with respect to the data, of the solution  $u$  of the Schrödinger equation (1.1) with non-homogeneous Dirichlet-boundary condition  $f \in H_0^{2,1}(\Sigma)$  and an initial data  $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ .

**A.1. Proof of Theorem 1.1.** We decompose the solution  $u$  of the Schrödinger equation (1.1) as  $u = u_1 + u_2$ , with  $u_1$  and  $u_2$  are respectively solutions to

$$\begin{cases} (i\partial_t + \Delta_A)u_1 = 0, & \text{in } Q \\ u_1(x, 0) = 0, & \text{in } \Omega \\ u_1(x, t) = f, & \text{on } \Sigma \end{cases}, \quad \begin{cases} (i\partial_t + \Delta_A + q)u_2 = -qu_1, & \text{in } Q \\ u_2(x, 0) = u_0, & \text{in } \Omega \\ u_2 = 0, & \text{on } \Sigma \end{cases}$$

Using the fact that  $f \in H_0^{2,1}(\Sigma)$ , we can see from [2][Theorem 1.1] that

$$(A.51) \quad u_1 \in C^1(0, T; H^1(\Omega)),$$

and

$$(A.52) \quad \|u_1\|_{C^1(0, T; H^1(\Omega))} \leq C \|f\|_{H^{2,1}(\Sigma)}.$$

Moreover, we have  $\partial_\nu u_1 \in L^2(\Sigma)$ , and we get a constant  $C > 0$  such that

$$(A.53) \quad \|\partial_\nu u_1\|_{L^2(\Sigma)} \leq C \|f\|_{H^{2,1}(\Sigma)}.$$

On the other hand, from [10][Lemma 2.1], we conclude the existence of a unique solution

$$(A.54) \quad u_2 \in C^1(0, T; L^2(\Omega)) \cap C(0, T; H^2(\Omega) \cap H_0^1(\Omega)),$$

that satisfies

$$(A.55) \quad \begin{aligned} \|u_2(\cdot, t)\|_{H_0^1(\Omega)} &\leq C \left( \|qu_1\|_{W^{1,1}(0, T; L^2(\Omega))} + \|u_0\|_{H_0^1 \cap H^2} \right) \\ &\leq C \left( \|u_0\|_{H_0^1 \cap H^2} + \|f\|_{H^{2,1}(\Sigma)} \right). \end{aligned}$$

Next, we consider a  $\mathcal{C}^2$  vector field  $N$  satisfying

$$N(x) = \nu(x), \quad x \in \Gamma, \quad |N(x)| \leq 1, \quad x \in \Omega.$$

Multiplying the second Schrödinger equation by  $N \cdot \nabla \bar{u}_2$  and integrating over  $Q = \Omega \times (0, T)$  we get

$$\begin{aligned} - \int_0^T \int_{\Omega} q u_1 N \cdot \nabla \bar{u}_2 dx dt &= i \int_0^T \int_{\Omega} \partial_t u_2 N \cdot \nabla \bar{u}_2 dx dt + \int_0^T \int_{\Omega} \Delta u_2 N \cdot \nabla \bar{u}_2 dx dt \\ &\quad + \int_0^T \int_{\Omega} (2iA \cdot \nabla + i \operatorname{div} A - |A|^2 + q) u_2 N \cdot \nabla \bar{u}_2 dx dt = I_1 + I_2 + I_3. \end{aligned}$$

By integrating with respect to  $t$  in the first term  $I_1$ , we get

$$\begin{aligned} I_1 &= i \int_{\Omega} \left[ u_2(x, T) N \cdot \nabla \bar{u}_2(x, T) - u_2(x, 0) N \cdot \nabla \bar{u}_2(x, 0) \right] dx \\ &\quad - i \int_0^T \int_{\Omega} N \cdot \nabla (u_2 \partial_t \bar{u}_2) dx dt + i \int_0^T \int_{\Omega} \partial_t \bar{u}_2 N \cdot \nabla u_2 dx dt. \end{aligned}$$

Therefore, bearing in mind that  $i \partial_t \bar{u}_2 = -q \bar{u}_1 - \overline{\Delta_A u_2} - q \bar{u}_2$ , we get

$$\begin{aligned} 2\Re I_1 &= i \int_{\Omega} \left[ u_2(x, T) N \cdot \nabla \bar{u}_2(x, T) - u_0 N \cdot \nabla \bar{u}_0 \right] dx - \int_0^T \int_{\Omega} \operatorname{div} N q u_2 \bar{u}_1 dx dt \\ &\quad - \int_0^T \int_{\Omega} \operatorname{div} N q |u_2|^2 dx dt + \int_0^T \int_{\Omega} \nabla_A (\operatorname{div} N u_2) \cdot \nabla_A \bar{u}_2 dx dt \\ &\quad - i \int_0^T \int_{\Gamma} u_2 \partial_t \bar{u}_2 d\sigma dt - \int_0^T \int_{\Sigma} \partial_{\nu} \bar{u}_2 (u_2 \operatorname{div} N) d\sigma dt. \end{aligned}$$

As the last term vanishes since  $u_2 = 0$  on  $\Sigma$ , we deduce from (A.55) that

$$|\Re I_1| \leq C \left( \|f\|_{H^{2,1}(\Sigma)}^2 + \|u_0\|_{H_0^1 \cap H^2}^2 \right).$$

On the other hand, by Green's Formula, we have

$$\begin{aligned} I_2 &= - \int_0^T \int_{\Omega} \nabla u_2 \nabla (N \cdot \nabla \bar{u}_2) dx dt + \int_0^T \int_{\Gamma} \partial_{\nu} u_2 (N \cdot \nabla \bar{u}_2) d\sigma dt \\ &= - \int_0^T \int_{\Omega} \nabla u_2 \cdot \nabla (N \cdot \nabla \bar{u}_2) dx dt + \int_0^T \int_{\Gamma} |\partial_{\nu} u_2|^2 d\sigma dt, \end{aligned}$$

So, we get

$$\begin{aligned} I_2 &= \int_0^T \int_{\Gamma} |\partial_{\nu} u_2|^2 d\sigma dt - \frac{1}{2} \int_0^T \int_{\Omega} \operatorname{div} (|\nabla u_2|^2 N) dx dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega} |\nabla u_2|^2 \operatorname{div} N dx dt - \int_0^T \int_{\Omega} DN (\nabla u_2, \nabla \bar{u}_2) dx dt. \end{aligned}$$

Thus, we have

$$\begin{aligned} I_2 &= \int_0^T \int_{\Gamma} |\partial_{\nu} u_2|^2 - \frac{1}{2} \int_0^T \int_{\Gamma} |\nabla u_2|^2 N \cdot \nu d\sigma dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega} |\nabla u_2|^2 \operatorname{div} N dx dt - \int_0^T \int_{\Omega} DN (\nabla u_2, \nabla u_2) dx dt. \end{aligned}$$

Next, using the fact that

$$|\nabla u_2|^2 = |\partial_{\nu} u_2|^2 + |\nabla_{\tau} u_2|^2 = |\partial_{\nu} u_2|^2, \quad x \in \Gamma,$$

where  $\nabla_\tau$  is the tangential gradient on  $\Gamma$ , we obtain

$$\begin{aligned} \Re I_2 &= \frac{1}{2} \int_0^T \int_\Omega |\partial_\nu u_2|^2 d\sigma dt + \frac{1}{2} \int_0^T |\nabla u_2|^2 \operatorname{div} N dx dt \\ &\quad - \int_0^T \int_\Omega DN(\nabla u_2, \nabla \bar{u}_2) dx dt. \end{aligned}$$

Moreover, by (A.55), it is easy to see that

$$|\Re I_3| \leq C \left( \|f\|_{H^{2,1}(\Sigma)}^2 + \|u_0\|_{H_0^1 \cap H^2}^2 \right),$$

so that, we deduce from the above statements that

$$\|\partial_\nu u_2\|_{L^2(\Sigma)} \leq C \left( \|f\|_{H^{2,1}(\Sigma)} + \|u_0\|_{H_0^1 \cap H^2} \right).$$

From the above reasoning, we conclude that  $u = u_1 + u_2 \in \mathcal{C}(0, T; H^1(\Omega))$ ,  $\partial_\nu u \in L^2(\Sigma)$  and we have

$$\|u(\cdot, t)\|_{H^1(\Omega)} + \|\partial_\nu u\|_{L^2(\Sigma)} \leq C \left( \|f\|_{H^{2,1}(\Sigma)} + \|u_0\|_{H_0^1 \cap H^2} \right).$$

## APPENDIX B. SOME FUNDAMENTAL STATEMENTS

In this section, we collect several technical results that are needed in the proof of the main results. We first introduce the following notations. Let  $P(D)$  be a differential operator with  $D = -i(\partial_t, \partial_x)$ . We denote by

$$\tilde{P}(\xi, \tau) = \left( \sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathbb{N}^n} |\partial_\tau^k \partial_\xi^\alpha P(\xi, \tau)|^2 \right)^{\frac{1}{2}}, \quad \xi \in \mathbb{R}^n, \tau \in \mathbb{R}.$$

For  $1 \leq p \leq \infty$ , we define the space

$$B_{p, \tilde{P}} = \{f \in S'(\mathbb{R}^{n+1}), \tilde{P}\mathcal{F}(f) \in L^p(\mathbb{R}^{n+1})\},$$

equipped with the following norm

$$\|f\|_{B_{p, \tilde{P}}} = \|\tilde{P}\mathcal{F}(f)\|_{L^p(\mathbb{R}^{n+1})}.$$

We finally denote by

$$B_{p, \tilde{P}}^{loc} = \{f \in S'(\mathbb{R}^{n+1}), \varphi f \in B_{p, \tilde{P}}, \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})\}.$$

We start by recalling some known results of Hörmander:

**Lemma B.1.** *Let  $u \in B_{\infty, \tilde{P}}$  and  $v \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})$ . Then, we have  $uv \in B_{\infty, \tilde{P}}$ , and*

$$\|uv\|_{\infty, \tilde{P}} \leq C \|u\|_{\infty, \tilde{P}},$$

where the positive constant  $C$  depends only on  $v$ ,  $n$  and the degree of  $P$ .

**Lemma B.2.** *Any differential operator  $P(D)$  admits a fundamental solution  $F \in B_{\infty, \tilde{P}}^{loc}$  satisfying  $\frac{F}{\cosh |(x, t)|} \in B_{\infty, \tilde{P}}$ . Moreover, it verifies*

$$\left\| \frac{F}{\cosh |(x, t)|} \right\|_{\infty, \tilde{P}} \leq C,$$

where  $C$  is a positive constant that depends only on  $n$  and the degree of  $P$ .

Our first goal in this section is to prove the following theorem:

**Theorem B.3.** *Let  $P \neq 0$  be a differential operator. Then for all  $k \in \mathbb{N}$ , there exists a linear operator*

$$E : L^2(0, T; H^k(\Omega)) \rightarrow L^2(0, T; H^k(\Omega)),$$

*such that:*

- (1)  $P(D)Ef = f$ , for any  $f \in L^2(0, T; H^k(\Omega))$ .
- (2) For any linear differential operator with constant coefficient  $Q(D)$  such that  $\frac{|Q(\xi, \tau)|}{\tilde{P}(\xi, \tau)}$  is bounded, we have  $Q(D)E \in B(L^2(0, T; H^k(\Omega)))$  and

$$\|Q(D)Ef\|_{L^2(0, T; H^k(\Omega))} \leq C \sup_{\mathbb{R}^{n+1}} \frac{|Q(\xi, \tau)|}{\tilde{P}(\xi, \tau)} \|f\|_{L^2(0, T; H^k(\Omega))},$$

where  $C$  depends only on the degree of  $P$ ,  $\Omega$  and  $T$ .

*Proof.* Let  $f \in L^2(0, T; H^k(\Omega))$ . There exists an extension operator

$$\begin{array}{ccc} S : L^2(0, T; H^k(\Omega)) & \longrightarrow & L^2(0, T; H^k(\mathbb{R}^n)) \\ f & \longmapsto & \tilde{f}, \end{array}$$

such that for all  $t \in (0, T)$ , we have  $\tilde{f}(\cdot, t)|_\Omega = f(\cdot, t)$ . Next, we introduce

$$\tilde{f}_0 = \begin{cases} \tilde{f}, & t \in (0, T), \quad x \in \mathbb{R}^n \\ 0, & t \notin (0, T), \quad x \in \mathbb{R}^n. \end{cases}$$

So, we have  $\tilde{f}_0|_Q = f$ . Let  $R > 0$  and  $V$  be a neighborhood of  $\overline{Q}$ . We consider  $\psi \in C_0^\infty(\mathbb{R}^{n+1})$  such that  $\psi|_V = 1$  and satisfying  $\text{supp } \psi \subset B(0, R) \subset \mathbb{R}^{n+1}$ . Let  $F$  be a fundamental solution of  $P$ . We consider the following operator

$$\begin{array}{ccc} E : L^2(0, T; H^k(\Omega)) & \longrightarrow & L^2(0, T; H^k(\Omega)) \\ f & \longmapsto & E(f) = (F * \psi \tilde{f}_0)|_Q \end{array}$$

Since  $P(D)(F * \psi \tilde{f}_0) = \psi \tilde{f}_0$ , then we clearly have

$$P(D)Ef = (\psi \tilde{f}_0)|_Q = f.$$

We turn now to proving the second point. For this purpose, we consider  $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$  such that  $\varphi = 1$  on a neighborhood of the closure of  $\{x - y, \quad x, y \in Q\}$ . We can easily verify that

$$(F * \psi \tilde{f}_0)|_Q = (\varphi F * \psi \tilde{f}_0)|_Q.$$

The last identity entails that for all  $\alpha \in \mathbb{N}^n$ , such that  $|\alpha| \leq k$ , we have

$$\begin{aligned} \|\partial^\alpha Q(D)Ef\|_{L^2(Q)} &= \|Q(D)\partial^\alpha(F * \psi \tilde{f}_0)\|_{L^2(Q)} \\ &= \|Q(D)\varphi F * \partial^\alpha(\psi \tilde{f}_0)\|_{L^2(Q)} \\ &\leq \|Q(D)\varphi F * \partial^\alpha(\psi \tilde{f}_0)\|_{L^2(\mathbb{R}^{n+1})} \\ &\leq \|\mathcal{F}(Q(D)\varphi F * \partial^\alpha(\psi \tilde{f}_0))\|_{L^2(\mathbb{R}^{n+1})} \\ &\leq \|Q(\xi, \tau)\mathcal{F}(\varphi F)\mathcal{F}(\partial^\alpha(\psi \tilde{f}_0))\|_{L^2(\mathbb{R}^{n+1})} \\ &\leq \|Q(\xi, \tau)\mathcal{F}(\varphi F)\|_{L^\infty(\mathbb{R}^{n+1})} \|\partial^\alpha(\psi \tilde{f}_0)\|_{L^2(\mathbb{R}^{n+1})} \\ &\leq \|Q(\xi, \tau)\mathcal{F}(\varphi F)\|_{L^\infty(\mathbb{R}^{n+1})} \|\partial^\alpha f\|_{L^2(Q)}. \end{aligned} \tag{B.56}$$

Using the fact that

$$Q(\xi, \tau) \mathcal{F}(\varphi F) = \frac{Q(\xi, \tau)}{\tilde{P}(\xi, \tau)} \tilde{P}(\xi, \tau) \mathcal{F}\left(\varphi \cosh |(x, t)| \frac{F}{\cosh |(x, t)|}\right),$$

we deduce from Lemma B.1 and Lemma B.2 that

$$(B.57) \quad \|Q(\xi, \tau) \mathcal{F}(\varphi F)\|_{L^\infty(\mathbb{R}^{n+1})} \leq C \sup_{(\xi, \tau) \in \mathbb{R}^{n+1}} \frac{|Q(\xi, \tau)|}{\tilde{P}(\xi, \tau)}.$$

Then from (B.56) and (B.57), we get

$$(B.58) \quad \|\partial^\alpha Q(D) E f\|_{L^2(Q)} \leq C \sup_{(\xi, \tau) \in \mathbb{R}^{n+1}} \frac{|Q(\xi, \tau)|}{\tilde{P}(\xi, \tau)} \|f\|_{L^2(0, T; H^k(\Omega))}, \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k.$$

Thus, we find that

$$\|Q(D) E f\|_{L^2(0, T; H^k(\Omega))} \leq C \sup_{\mathbb{R}^{n+1}} \frac{|Q(\xi, \tau)|}{\tilde{P}(\xi, \tau)} \|f\|_{L^2(0, T; H^k(\Omega))},$$

which completes the proof of the lemma.  $\square$

Finally, we establish the following statement:

**Lemma B.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a simply connected domain, and let  $A \in \mathcal{C}^2(\Omega, \mathbb{R}^n)$  be such that  $A|_\Gamma = 0$ . Then, for  $p > n$ , there exists a function  $\varphi \in \mathcal{C}^3(\Omega)$  such that  $\varphi|_\Gamma = 0$  and  $A' \in W^{1,p}(\Omega, \mathbb{R}^n)$ , satisfying  $A = A' + \nabla \varphi$ ,  $A' \wedge \nu = 0$ , and  $\operatorname{div} A' = 0$ . Moreover, there exists a constant  $C > 0$ , such that*

$$(B.59) \quad \|A'\|_{W^{1,p}(\Omega)} \leq C \|\operatorname{curl} A'\|_{L^p(\Omega)}.$$

*Proof.* Let  $\varphi$  be the solution of the following problem

$$(B.60) \quad \begin{cases} \Delta \varphi = \operatorname{div} A, & \text{in } \Omega \\ \varphi = 0, & \text{in } \Gamma. \end{cases}$$

Then, setting  $A' = A - \nabla \varphi$ , using the fact that  $A|_\Gamma = \varphi|_\Gamma = 0$ , one gets

$$A' \wedge \nu = A \wedge \nu - \nabla \varphi \wedge \nu = 0, \quad \text{and} \quad \operatorname{div} A' = 0.$$

In order to prove (B.59), we argue by contradiction. We assume that for all  $k \geq 1$  there exists a non-null  $\widetilde{A}'_k \in W^{1,p}(\Omega)$  such that

$$(B.61) \quad \|\widetilde{A}'_k\|_{W^{1,p}(\Omega)} \geq k \|\operatorname{curl} \widetilde{A}'_k\|_{L^p(\Omega)}.$$

We set  $A'_k = \frac{\widetilde{A}'_k}{\|\widetilde{A}'_k\|_{W^{1,p}(\Omega)}}$ . Then we have  $\|A'_k\|_{W^{1,p}(\Omega)} = 1$  and  $k \|\operatorname{curl} A'_k\|_{L^p(\Omega)} \leq 1$ . In view of the weak

compactness theorem, there exists a subsequence of  $(A'_k)_k$  such that  $A'_k \rightharpoonup A'$  in  $W^{1,p}(\Omega)$ . Using the fact that  $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ , we deduce that  $A'_k \rightarrow A'$  in  $L^p(\Omega)$ . As a consequence, we have

$$\|A'\|_{W^{1,p}(\Omega)} = 1 \quad \text{and} \quad \|\operatorname{curl} A'\|_{L^p(\Omega)} = 0.$$

This entails that there exists  $\eta \in W^{1,p}(\Omega)$  such that  $A' = \nabla \eta$ . Then, using the fact that  $\operatorname{div} A' = 0$  and  $A' \wedge \nu = 0$ , we deduce that there exists a constant  $\lambda \in \mathbb{R}$  such that

$$\begin{cases} \Delta \eta = 0, & \text{in } \Omega \\ \eta = \lambda, & \text{in } \Gamma. \end{cases}$$

Finally, using the fact that  $\Omega$  is a simply connected domain we conclude that  $\eta = \lambda$  in  $\overline{\Omega}$ . This entails that  $A' = 0$  and contradicts the fact that  $\|A'\|_{W^{1,p}(\Omega)} = 1$ .  $\square$

As a consequence of Lemma B.4, we have the following result

**Lemma B.5.** *Let  $\Omega \subset \mathbb{R}^n$  be a simply connected domain, and let  $A \in \mathcal{C}^2(\Omega, \mathbb{R}^n)$  such that  $A|_{\Gamma} = 0$ . If we further assume that  $\operatorname{div} A = 0$ , then the following estimate*

$$\|A\|_{W^{1,p}(\Omega)} \leq C \|\operatorname{curl} A\|_{L^p(\Omega)},$$

*holds true for some positive constant  $C$  which is independent of  $A$ .*

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## REFERENCES

- [1] J. Apraiz, L. Escauriaza, *Null-control and measurable sets*, ESAIM: Control, Optimisation and calculus of variations, 19, (2013) 239-254, .
- [2] M. Bellassoued, *Stable determination of coefficients in the dynamical Shrödinger equation in a magnetic field*, arXiv:1510.04247v1.
- [3] M. Bellassoued, I. Ben Aïcha, *Stable determination outside a cloaking region of two time-dependent coefficients in an hyperbolic equation from Dirichlet to Neumann map*, arXiv:1605.03466v1.
- [4] M. Bellassoued, H. Benjoud, *Stability estimate for an inverse problem for the wave equation in a magnetic field*, Appl. Anal. 87, No. 3, 277-292 (2008).
- [5] M. Bellassoued, M. Choulli, *Stability estimate for an inverse problem for the magnetic Schrödinger equation from the Dirichlet-to-Neumann map*, Journal of Functional Analysis, 258, 1 (2010), 161-195.
- [6] M. Bellassoued, D. Jellali, M. Yamamoto, *Stability estimate for the hyperbolic inverse boundary value problem by local Dirichlet-to-Neumann map*, J. Math. Anal. Appl. 343 (2008), no. 2, 1036-1046.
- [7] M. Bellassoued, D. Dos Santos Ferreira, *Stable determination of coefficients in the dynamical anisotropic Schrödinger equation from the Dirichlet-to-Neumann map*, Inverse Problems 26 (2010) 125010 (30pp).
- [8] I. Ben Aïcha, *Stability estimate for a hyperbolic inverse problem with time-dependent coefficient*, Inverse Problems 31 (2015) 125010 (21pp).
- [9] H. Ben Joud, *Stability estimate for an inverse problem for the Schrödinger equation in a magnetic field from partial boundary measurements*, Inverse Problems (2009).
- [10] M. Choulli, Y. Kian, E. Soccorsi, *Stable determination of time-dependent scalar potential from boundary measurements in a periodic quantum waveguide*, SIAM J. Math. Anal., 47(6) (2015) (23 pp).
- [11] L. Hörmander, *The Analysis of linear partial differential operators*, Vol II, Springer-Verlag, Berlin, Heidelberg, 1983.
- [12] M. M Lavrent'ev, V. G. Romanov, Shishat·skii *Ill posed Problems of Mathematical Physics and Analysis*
- [13] Rakesh and W. Symes, *Uniqueness for an inverse problem for the wave equation*, Comm. Partial Diff. Equations 13 (1988), 87-96.
- [14] A. Ramm and J. Sjöstrand, *An inverse problem of the wave equation*, Math. Z. 206 (1991), 119-130.
- [15] R. Salazar *Determination of time-dependent coefficients for a hyperbolic inverse problem*, Inverse Problems (2013) 29095015.
- [16] M. Salo, *Semiclassical pseudodifferential calculus and the reconstruction of a magnetic field*, Comm. P.D. E, 31 (2006), No 11, 1639-1666.
- [17] M. Salo, *Inverse problem for nonsmooth first order perturbations of the Laplacian*, Ann. Acad. Sci. Fenn. Math. Diss, (2004) 139.
- [18] P. Stefanov, *Uniqueness of the multi-dimentionnal inverse scattering problem for time-dependent potentials*, Math. Z. (1994) 201 541-59
- [19] Z. Sun, *An inverse boundary value problem for the Schrödinger operator with vector potentials*, Trans. Amer. Math. Soc, Vol.338.No.2, (1992),953-969.
- [20] Y. Kian, *Stability in the determination of a time dependent coefficient for wave equations from partial data*, Mathematical analysis and applications Vol.436, (2016), 408 428
- [21] G. Eskin, *Inverse problems for the Schrödinger equations with time-dependent electromagnetic potentials and the Aharonov-Bohm effect*, J. Math. Phys. 49 (2008), no.2, 022105, 18 pp.
- [22] G. Eskin, *Inverse hyperbolic problems with time-dependent coefficients*, Comm. Partial Diff Equations, (2007) 32, 11, 1737-1758.
- [23] V. Isakov, *An inverse hyperbolic problem with many boundary measurements*, Comm. Partial Diff Equations, (1991) 16, 1183-1195.

- [24] Z. Sun, *An inverse boundary value problem for the Schrödinger operator with vector potentials*, Trans. Amer. Math. Soc., Vol. 338. No. 2, (1992), 953-969.
- [25] L. Tzou, *Stability estimates for coefficients of magnetic Shrödinger equation from full and partial boundary measurements*, Comm. Partial Differential Equations, 11 (2008), 1911-1952.
- [26] S. Vessella *A continuous dependence result in the analytic continuation problem*.
- [27] A. Waters, *Stable determination of X-ray transforms of time-dependent potentials from the dynamical Dirichlet-to-Neumann map*, Comm. Partial Diff. Equations, (2014) 39, 12.

I. BEN AÏCHA. UNIVERSITY OF AIX-MARSEILLE, 58 BOULEVARD CHARLES LIVON, 13284 MARSEILLE, FRANCE. & UNIVERSITY OF CARTHAGE, FACULTY OF SCIENCES OF BIZERTE, 7021 JARZOUNA BIZERTE, TUNISIA. & LAMSIN, NATIONAL ENGINEERING SCHOOL OF TUNIS, B.P. 37, 1002 TUNIS, TUNISIA

*E-mail address:* ibtissem.benaicha@enit.utm.tn